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Abstract: We introduce automata over the $(\max, +)$ algebra: this formalism encompasses in particular $(\max, +)$ linear systems and a subclass of stochastic timed event graphs. Performance evaluation is considered in the worst, mean, and optimal cases. A simple algebraic reduction is provided for the worst case. The later cases are reduced to projective finiteness properties of semigroups of matrices. The mean performance is given by the Kolmogorov equation of a Markov chain in the projective space. The optimal performance is given by an analogous Hamilton-Jacobi-Bellman equation.

Key-words: Discrete Event Systems, Automata, Rational Series, Lyapunov Exponents, Kolmogorov Equation, Hamilton-Jacobi-Bellman Equation.

(Résumé : tsvp)

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Évaluation de Performance d'Automates Temporisés

Résumé : Nous étudions les automates sur le semianneau $(\max, +)$. Ceux-ci recouvrent en particulier les systèmes $(\max, +)$-linéaires et une sous-classe de graphes d'événements temporisés stochastiques. On cherche à calculer certaines mesures de performance: dans le pire, le meilleur des cas, ainsi qu'en moyenne. Le pire des cas se traite par un simple calcul algébrique. Quant aux deux autres cas, on se ramène à des propriétés de finitude projective des semigroupes de matrices. La performance dans le cas moyen est donné par l'équation de Kolmogorov d'un chaîne de Markov dans un espace projectif. Pour le meilleur des cas, une équation de Hamilton-Jacobi-Bellman joue un rôle analogue.

Mots-clé : Systèmes à Événements Discrets, Automates, Séries Rationnelles, Exposants de Lyapunov, Équation de Kolmogorov, Équation de Hamilton-Jacobi-Bellman.

1 Introduction

Among the algebraic tools previously introduced in the study of discrete events systems, we may distinguish two separated formalisms:

- automata (or rational languages), in the framework of supervisory control [30].
- $(\max, +)$ algebra [10].

The first approach was (at the beginning) purely logic (it did not include time modelization). On the other hand, $(\max, +)$-systems focus on the time-behavior, at the price of an important loss of generality by comparison with automata. Quite recently, some extension of supervisory control to timed discrete event systems has been made [8]. Roughly speaking, this amounts to restricting the language recognized by an automaton according to some time constraints. Here, we propose a different generalization, which is better explained by an algebraic argument. Let Σ denote a finite alphabet. A rational language can be identified to its characteristic rational boolean series[1]

$$s = \bigoplus_{w \in \Sigma^*} (s|w)\, w \in \mathbb{B}\langle\langle \Sigma \rangle\rangle \tag{1}$$

where $\mathbb{B} = \{\varepsilon, e\}$ denotes the boolean semiring. On the other hand, a discrete causal SISO $(\max, +)$ linear system with finite dimension can be represented by a $(\max, +)$ rational series

$$s = \bigoplus_{n \in \mathbb{N}} s_n X^n \in \mathbb{R}_{\max}[[X]] \ , \tag{2}$$

where $\mathbb{R}_{\max}$ denotes the semiring[2]. $(\mathbb{R} \cup \{-\infty\}, \max, +)$. Since $\mathbb{B}$ is a subsemiring of $\mathbb{R}_{\max}$, the "least upper bound" of both formalisms is provided by $(\max, +)$-*rational series*, which write

$$s = \bigoplus_{w \in \Sigma^*} (s|w)\, w \in \mathbb{R}_{\max}\langle\langle \Sigma \rangle\rangle \ . \tag{3}$$

Equivalently, s is the series recognized by a $(\max, +)$ automaton[2]. Then, two questions naturally arise: (i) is the class of systems covered by (3) strictly more interesting than the previous ones; (ii) do rational series of the form (3) enjoy new algebraic properties allowing performance evaluation of certain discrete event systems? We claim that the answer to both questions is positive. The study of the power of modelization of $(\max, +)$ rational series and the building of a linear system theory in this context requires a complete treatment, which is beyond the scope of this paper: this will be done elsewhere. Here, we shall just give some idea of the class of phenomena dealt with, in order to convince the reader of the relevance of $(\max, +)$ automata for modeling discrete event dynamic systems. We consider here the algebraic problems related with performance evaluation of timed automata. We give some measures of performance in the worst, mean and optimal cases. We show that the mean case and optimal case evaluation can be obtained from projective finiteness properties of semigroups of matrices, leading to a Kolmogorov equation (for the mean case) and an Hamilton-Jacobi-Bellman equation (for the optimal case). It turns out that not only the mean case evaluation is interesting in itself, but it also solves the problem of the computation of the Lyapunov exponents for a subclass of stochastic timed event graphs [2]. Another interesting problem also can be dealt with using timed automata. We consider $(\max, +)$-linear systems of the form

[1] $(s|w)$ denotes the coefficient of the word w, equal to the zero "ε" if $w \notin L$ and to the unit "e" if $w \in L$. Formal series will be presented with more details in the following sections.

[2] precisely defined below.

$x(k) = A(k)x(k-1), x(0) = x_0$ where the matrices $A(k)$ can be chosen among a finite set of matrices in order to minimize some final time, written $cx(N)$. This problem reduces to the optimal case evaluation of timed automata. Indeed, these two last points can be seen as an algebraic generalization of a markovian technique due to Olsder [29] (see also [2], chapter 9). We conclude the paper by giving an algebraic proof of Baccelli's lower bound for the Lyapunov exponent, using spectral properties of Hadamard products of matrices in the $(\max, +)$ algebra.

2 Timed Automata

2.1 Example

Let us consider a storage with a capacity of two units. The two following events are possible:

a	a part is added to the stock
b	a part is taken out.

This system is represented by the (conventional) automaton of Figure 1 over the alphabet $\Sigma = \{a, b\}$. Node 0 represents the state "0 part in stock", node 1 "1 part in stock", etc. First, the storage is empty. Recall that Σ^* denotes the free monoid over Σ (i.e. the set of words, equipped with the concatenation product). We interpret each word $w = a_1 \dots a_n \in \Sigma^*$ as a sequence of events. We consider the situation where the transitions of the automaton take at least some given times. For instance, we assume that the transition $1 \xrightarrow{a} 2$ takes at least 4 unit of time, and we write $1 \xrightarrow{4a} 2$. This leads us to the following definitions. A *dater* is a map

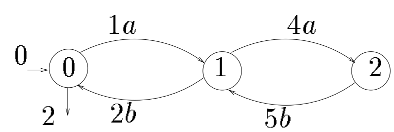

Figure 1: Storage with capacity of two units

$$x : \Sigma^* \to \mathbb{R} \cup \{-\infty\} .$$

A *timed automata* over the alphabet Σ is a directed graph with a set of nodes (or *states*) Q and three kind of arcs. (i) internal arcs, labeled with letters of Σ and valued with durations (ii) input arcs, valued with durations (but not labeled) (iii) output arcs, dual of input arcs. With each state $q \in Q$ is associated a dater x_q. $x_q(w)$ represents the "completion time of the event w at node q". It is computed according to the following rule.

(a) Let $T_{q,a,q'}$ denote the valuation (in time units) of the arc $q \mapsto q'$ with label a (if such an arc does not exist, we set $T_{q,a,q'} = -\infty$). Then, the *earliest* behavior is described by the following equations:

$$x_{q'}(wa) = \max_q [x_q(w) + T_{q,a,q'}] \tag{4}$$

In other other words, the event wa can be completed at node q' if for all possible transitions $q \xrightarrow{a} q'$, a time of $T_{q,a,q'}$ has been spent since the event w was complete at node q.

(b) Some initial conditions are given by the input arcs. Let α_q denote the valuation of the input arc at node q ($\alpha_q = -\infty$ if there is not input arc at q). We set

$$\forall q \in Q, \quad x_q(e) = \alpha_q \quad . \tag{5}$$

That is, α represents the date of the "zero" event. For dual reasons (whose importance will appear soon), we introduce a vector of final durations $\beta \in (\mathbb{R} \cup \{-\infty\})^Q$: β_q is equal to the valuation of the output arc at node q. The "final dater" y is defined by

$$y(w) = \max_{q \in Q} [\theta_q(w) + \beta_q] \quad . \tag{6}$$

For most applications, $y(w)$ will represent some "global" date of completion of the task w, For the automaton of Figure 1, we have $Q = \{0, 1, 2\}$ and

$$T_{1,a,2} = 4, \quad T_{1,b,2} = -\infty, \ldots$$

$$\alpha_q = \begin{cases} 0 & \text{if } q = 0 \\ -\infty & \text{otherwise} \end{cases}$$

$$\beta_q = \begin{cases} 2 & \text{if } q = 0 \\ -\infty & \text{otherwise} \end{cases}$$

2.2 $(\max, +)$-automata

We next show that the earliest behavior y defined here above admits a very familiar algebraic transcription. Indeed, this is nothing but a specialization of the notion of K-automata [13] or equivalently, of K-rational series [6] to the $(\max, +)$ semiring. It should be noted that $(\min, +)$-automata (dual of $(\max, +)$ automata) have already been used by Simon [32] and Hashiguchi [21] in connection with some distance and complexity problems. See also Mascle [27] and Krob [24].

Recall that the $(\max, +)$ algebra[28, 2, 11, 19, 37] is by definition the set $\mathbb{R} \cup \{-\infty\}$ together with the laws max (denoted by $\oplus$) and $+$ (denoted by $\otimes$). E.g. $2 \otimes 1 = 3$, $2 \oplus -1 = 2$. The element $\varepsilon \stackrel{\text{def}}{=} -\infty$ satisfies $\varepsilon \oplus x = x$ and $\varepsilon \otimes x = \varepsilon$ (ε acts as a zero). The element $e \stackrel{\text{def}}{=} 0$ satisfies $e \otimes x = x$ (e is the unit). The main discrepancy with conventional algebra is that $x \oplus x = x$. We shall denote $\mathbb{R}_{\max} \stackrel{\text{def}}{=} (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ this structure. $\mathbb{R}_{\max}$ is a special instance of *dioid* (semiring whose addition is idempotent).

We first notice that with the dioid notation, (4) now writes

$$x_{q'}(wa) = \bigoplus_q x_q(w) \otimes T_{q,a,q'} \quad .$$

In other words, introducing the matrix

$$\mu(a) \in \mathbb{R}_{\max}^{Q \times Q}, \quad \mu(a)_{qq'} \stackrel{\text{def}}{=} T_{q,a,q'} \tag{7}$$

we get

$$x(wa) = x(w) \otimes \mu(a)$$

where $\otimes$ denotes the matrix product in the $(\max, +)$ algebra. In the following, we shall as usual omit the sign $\otimes$. After a straightforward induction on the length of w, we see that the earliest internal daters $(x_q)_{q \in Q}$ and the earliest final dater y are given by

$$x(w) = \alpha\mu(w), \quad y(w) = x(w)\beta \tag{8}$$

where μ is the unique morphism $\Sigma^* \to \mathbb{R}_{\max}^{Q\times Q}$ extending (7). That is, if $w = a_1 \dots a_n$, $\mu(w) = \mu(a_1)\dots\mu(a_n)$.

Therefore, a (finite[3]) $(\max,+)$ automaton can be equivalently described by a triple (α,μ,β), with $\alpha,\beta \in \mathbb{R}_{\max}^{Q}, \mu : \Sigma^* \to \mathbb{R}_{\max}^{Q\times Q}$ and Q finite.

2.3 Some applications of $(\max,+)$-rational series

We next recall some very classical notions about formal series (a good reference is [6]). A formal series over the alphabet Σ with coefficients in $\mathbb{R}_{\max}$ is a formal sum:

$$y = \bigoplus_{w\in\Sigma^*} (y|w)w \tag{9}$$

where $(y|w) \in \mathbb{R}_{\max}$ denotes the coefficient of y at the word w. Formal series are equipped with the componentwise sum and Cauchy product:

$$\begin{aligned} (y \oplus y'|w) &\stackrel{\text{def}}{=} (y|w) \oplus (y'|w), \\ (y \otimes y'|w) &\stackrel{\text{def}}{=} \bigoplus_{uv=w} (y|u) \otimes (y'|v) \ . \end{aligned}$$

The notation $\mathbb{R}_{\max}\langle\langle\Sigma\rangle\rangle$ for this dioid is standard. The subdioid of polynomials (such that $(y|w) \neq \varepsilon$ for a finite number of w) will be denoted by $\mathbb{R}_{\max}\langle\Sigma\rangle$. We shall denote by ε the neutral for sum $((\varepsilon|w) \stackrel{\text{def}}{=} -\infty, \forall w)$, and by e the unit

$$(e|w) \stackrel{\text{def}}{=} \begin{cases} 0 = e_{\mathbb{R}_{\max}} & \text{if } w = e \text{ (empty word)} \\ -\infty = \varepsilon_{\mathbb{R}_{\max}} & \text{otherwise.} \end{cases}$$

The scalar product of a series s and a polynomial t is defined by

$$(s|t) \stackrel{\text{def}}{=} \bigoplus_{w\in\Sigma^*} (s|w)(t|w) \tag{10}$$

which accounts for the notation "$(s|w)$" for the coefficient of the word w. In the following, we shall identify the dater map $y : \Sigma^* \to \mathbb{R}_{\max}, w \mapsto y(w)$ with the formal series

$$y = \bigoplus_{w\in\Sigma^*} y(w)\, w$$

(thus, $(y|w) = y(w)$). We say that a series y is *recognizable* if there exists a finite automaton (α,μ,β). such that

$$y = \bigoplus_{w\in\Sigma^*} \alpha\mu(w)\beta w \ . \tag{11}$$

Then, y is the series *recognized* by the automaton. Observe that sums of the form (11) or (9) are only formal (no infinite sum of scalars in involved). In the following, we shall need dealing with infinite sums of formal series also. In the standard theory of formal series over semirings, this is usually done by introducing the natural ultrametric structure over the formal series [6]. In the dioid's case, it is not more complicated and it is more general to define infinite sums as upper bounds with respect to a canonic order relation. More precisely, we introduce the *natural order* relation $\preceq$:

$$a \preceq b \iff a \oplus b = b \ .$$

[3] By finite automaton, we mean an automaton with a finite number of states

In $\mathbb{R}_{\max}$, this coincides with the usual order, since $a \leq b \iff \max(a,b) = b$. More generally, it is easily checked that $a \oplus b$ is equal to the least upper bound of a, b for the order $\preceq$. This allows us to set

$$\bigoplus_{x \in X} x \stackrel{\text{def}}{=} \sup X$$

for any set X admitting a least upper bound: this coincides with addition in the case of finite sums. Moreover, it is plain that in $\mathbb{R}_{\max}$ (and hence, in $\mathbb{R}_{\max}\langle\langle\Sigma\rangle\rangle$), the infinite distributivity holds, i.e. for all X admitting a least upper bound:

$$\left(\bigoplus_{x \in X} x\right) y = \bigoplus_{x \in X} xy, \quad y\left(\bigoplus_{x \in X} x\right) = \bigoplus_{x \in X} yx$$

The *star* of the series $y \in \mathbb{R}_{\max}\langle\langle\Sigma\rangle\rangle$ is by definition

$$y^* = e \oplus y \oplus y^2 \oplus y^3 \dots \tag{12}$$

It is easy to see that this sum converges in $\mathbb{R}_{\max}\langle\langle\Sigma\rangle\rangle$ iff $(y|e) \preceq e$. A series is *rational* iff it is obtained by a finite number of sums, products, and stars, starting from polynomials. Let us recall that when Σ is finite, the Kleene-Schützenberger theorem [6, 13] asserts that a series is recognizable iff it is rational. Since we are basically interested with systems allowing only a finite number of elementary events, we shall always assume Σ finite, and we can say that our object of study indeed concerns $(\max,+)$-rational series.

2.3.1 Example For the automaton of Figure 1, we have:

$$\alpha = \begin{bmatrix} e & \varepsilon & \varepsilon \end{bmatrix}, \; \beta = \begin{bmatrix} 2 \\ \varepsilon \\ \varepsilon \end{bmatrix},$$

$$\mu(a) = \begin{bmatrix} \varepsilon & 1 & \varepsilon \\ \varepsilon & \varepsilon & 4 \\ \varepsilon & \varepsilon & \varepsilon \end{bmatrix}, \; \mu(b) = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon \\ 2 & \varepsilon & \varepsilon \\ \varepsilon & 5 & \varepsilon \end{bmatrix} .$$

Let us compute $(x|w)$ in a particular case:

$$(x|ab) = \alpha\mu(ab) = \alpha\mu(a)\mu(b) = \begin{bmatrix} 3 & \varepsilon & \varepsilon \end{bmatrix},$$

$$(y|ab) = (x|ab)\beta = 5 \ .$$

More generally, it is easily shown that the series y admits the following rational expression

$$\begin{aligned} y &= 2(3a(9ab)^*b)^* \\ &= 2 \oplus 5ab \oplus 8abab \oplus 14aabb \oplus \dots \end{aligned}$$

It is important to notice that contrarily to the mainstream of supervisory control literature [30], $(\max,+)$-automata are no longer required to be *deterministic*[4]. Indeed, the traditional word "non

[4] Recall that the automaton (Q, α, μ, β) is deterministic if there is a single initial state q (such that $\alpha_q \neq \varepsilon$), and for all $(q, x) \in Q \times \Sigma$, $\#\{q' \in Q \mid \mu(x)_{qq'} \neq \varepsilon\} \leq 1$ (there is a most one internal arc with a given label x outgoing from node q)

deterministic" is misleading in the $(\max,+)$ case. For instance, consider the $(\max,+)$ automaton over the alphabet $\Sigma=\{a\}$ given in Figure 2. The earliest dater satisfies:

$$(x_1|wa)=\max(2+(x_0|w),1+(x_0|w)) \ . \tag{13}$$

In other words, the two parallel "non deterministic" edges $0\xrightarrow{a}1$ amount to starting *simultaneously* two tasks from node 0 to node 1: node 1 waits for the completion of both tasks. That is, "non deterministic edges" represent *synchronization* phenomena.

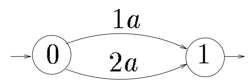

Figure 2: Synchronization of nondeterministic edges

Now, we think that it is worth comparing $(\max+)$-automata to more classical models.

(i) $(\max,+)$-linear systems. It is well known [2, 10] that a subclass of DEDS subject to saturation and synchronization constraints writes linearly in the $(\max,+)$-algebra[5]:

$$\begin{aligned} x(n+1) &= x(n)A, \quad x(0)=b,\\ y(n) &= x(n)c \ . \end{aligned} \tag{14}$$

It should be clear that (14) is a specialization of (8) when Σ is reduced to a single letter (set $\Sigma=\{x\}$, $\alpha=b$, $\mu(x)=A$, $\beta=c$). It should be noted that the series $\bigoplus_w \alpha\mu(w)\beta w$ reduces to the usual $\bigoplus_n bA^n cx^n$.

(ii) Stochastic timed event graphs (see Baccelli [3, 1], Olsder [29] and [2]). These systems write in the $(\max,+)$-algebra:

$$x(k)=x(0)A(1)\dots A(k-1)A(k) \tag{15}$$

where the $A(i)$ are random $n\times n$-matrices. When the random variables $A(i)$ only take a finite number of distincts values $A_1,\dots,A_p$, (15) writes $\alpha\mu(w)$, where w is a random word of length k. More precisely, let $\Sigma=\{A_1,\dots,A_p\}$, set $\alpha=x(0)$, and let μ be the unique morphism $\Sigma^*\to\mathbb{R}_{\max}^{n\times n}$ such that $\forall i=1,\dots,p$, $\mu(A_i)=A_i$. Then the word $w=a_1\dots a_k\in\Sigma^*$ represents the trajectory "$A(1)=a_1, A(2)=a_2,\dots,A(k)=a_k$", and $\alpha\mu(w)=x(k)$

(iii) Automata. Replace the non ε entries of α,μ,β by the unit. Then, we get an (α',μ',β') representation of a series y' such that

$$y'=\bigoplus_{w\in\Sigma^*}\alpha'\mu'(w)\beta' w\in\mathbb{B}\langle\langle\Sigma\rangle\rangle \ .$$

Thus, the series recognized by (α',μ',β') can be identified with a rational language.

(iv) $(\min,+)$-bilinear systems. We obtain the counterpart of Fliess generating series [14]. Consider the following systems of equations in the semiring $\mathbb{R}_{\min}\stackrel{\text{def}}{=}(\mathbb{R}\cup\{+\infty\},\min,+)$ (isomorphic to $\mathbb{R}_{\max}$):

$$\begin{aligned} x(0) &= b\\ x(t+1) &= x(t)A_0\oplus\textstyle\bigoplus_{i=1}^k x(t)A_iu_i(t)\\ y(t) &= x(t)c \ . \end{aligned} \tag{16}$$

[5]Here, writing systems from left to right is not mere provocation. Otherwise, it should be necessary to give a dual definition of automata in which words are read from right to left.

where the u_i denote some scalar inputs, $y(t)$ the scalar output of the system, and A_i, b, c are matrices and vectors of appropriate sizes. The system (16) represents a subclass of Timed Petri Nets where places of the type P_1 of Figure 3 are allowed (the *counter function* $x_i(t)$ denotes the number of firings of the transition labeled x_i up to time t). For instance, for the graph of Figure 3, we have

$$x_2(t+1) = 1 + u_1(t) + x_1(t) = 1 \otimes u_1(t) \otimes x_1(t)$$
$$x_1(t+1) = 2 \otimes x_2(t), \quad y(t) = x_2(t) \ .$$

This is clearly of the form (16): More generally, it should be clear that the class of systems modelizable by (16) consists in an extension of timed event graphs in which some tokens (corresponding to the inputs u_i) can be added at certain times in some given places. Let us

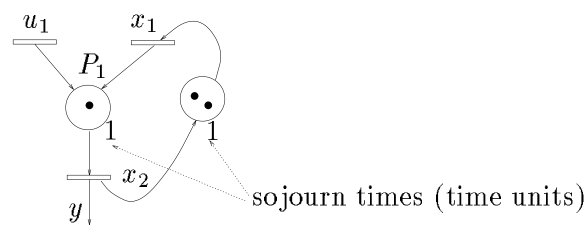

Figure 3: A $(\min,+)$-bilinear system

introduce as in the classical case an alphabet $\Sigma = \{a_1, \ldots, a_k\}$ (as many letters as controls) and the rational series:

$$(s|w) = b\mu(w)c, \ \text{ with } \mu(a_i) = A_i \ .$$

The *evaluation* of the word $w = a_{i_1} \ldots a_{i_p} \in \Sigma^p$ at u is defined by

$$w|_u = u_{i_p}(0) \ldots u_{i_2}(p-2) u_{i_1}(p-1) \ .$$

Then it is easily checked that

$$y(p) = \bigoplus_{w \in \Sigma^p} (s|w) w|_u$$

(the output is obtained by "evaluating" the rational series s). This suggests that $(\min,+)$-rational series will play for (min,+)-bilinear systems a role as important as rational generating series for conventional non-linear systems.

We conclude this introductive part by discussing in detail an example which illustrates the properties that we shall prove later on.

2.4 Example: workshop with two production regimes

We consider a workshop with two machines processing three types of parts. The workshop admits two configurations:

(a) a part of type 2 is processed by machine 2 during 5 units of time, another part of type 1 is processed by machine 1 during 3 units of time, the previous type 2 part is conveyed from machine 2 to machine 1 (which takes 4 units of time), then, it is instantaneously assembled to the part produced by machine 1 and exits the system.

(b) a part of type 3 is processed during 10 units of time on machine 1. Machine 2 remains idle.

We represent a behavior of the system by a word $w \in \{a,b\}^*$. E.g. $aaab$ means "3 working periods of type (a) followed by a period of type (b)". Let $(x_1|w)$ (resp. $(x_2|w)$) denote the date associated with the completion of the sequence w on machine 1 (resp. 2). We assume the initial condition $(x|e) = [e,e] = \alpha$. We are interested in computing $(y|w) = (x_1|w)$ which corresponds to the date at wich the last part produced by the event sequence w exits the system. It is not difficult to realize that y is recognized by the following linear representation

$$\mu(a) = \begin{bmatrix} 3 & \varepsilon \\ 4 & 5 \end{bmatrix}, \quad \mu(b) = \begin{bmatrix} 10 & \varepsilon \\ \varepsilon & e \end{bmatrix},$$
$$\alpha = \begin{bmatrix} e & e \end{bmatrix}, \beta = \begin{bmatrix} e \\ \varepsilon \end{bmatrix} \quad ,$$

i.e. $(y|w) = \alpha\mu(w)\beta$. This automaton is shown on Figure 4. Due to the triangular form of μ, we

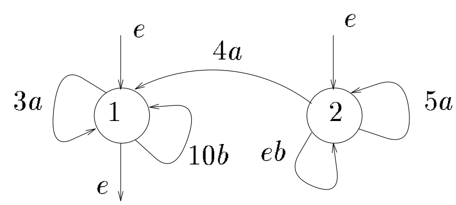

Figure 4: Workshop with two working regimes

obtain after a straightforward induction for $w \neq e$

$$(y|w) = 3^{|w|_a} \otimes 10^{|w|_b} \oplus \bigoplus_{uav=w} 4 \otimes 5^{|u|_a} \otimes 3^{|v|_a} \otimes 10^{|v|_b} \tag{17}$$

where $|z|_x$ denotes the number of occurrences of the letter x in the word z. Observe that, for instance, $4 \otimes 5^{|u_a|}$ in the dioid algebra is equal to $4 + 5 \times |u_a|$ in the usual algebra. The first coefficients of y are the following:

$$(y|a) = 4, (y|b) = \varepsilon,\ (y|a^2) = 9, (y|ab) = 14\ldots$$

Some interesting features appear when considering sub-behaviors of the systems. That is, instead of computing $(y|w)$ for all w, we assume that w belongs to a language L. Assume for instance a periodic behavior of the form $L_l = (a^l b)^*$ (that is, 1 period of type (b) occurs every l periods of type (a)). Then, it is not too difficult to obtain from (17):

$$(y|(a^l b)^i) = (2 \times l - 1) + (3 \times l + 10) \times i \ . \tag{18}$$

This formula admits a simple graphical interpretation. It means that a path with label $(a^l b)^i$ and with maximal weight from the input arcs to the output arc of the automaton of Figure 4 has the following form:

$$2 \xrightarrow{a^{l-1}} 2 \xrightarrow{a} 1 \xrightarrow{b(a^l b)^{i-1}} 1 \ .$$

Let us introduce

$$y_k \stackrel{\text{def}}{=} (y|(a^l b)^* \cap \Sigma^k)$$

(y_k represents the duration of the first k events). We get from (18) the following periodicity property

$$y_{k+c} = \lambda^c \otimes y_k = c \times \lambda + y_k \tag{19}$$

with $c = l + 1$ and

$$\lambda = \frac{3 \times l + 10}{l+1} \ . \tag{20}$$

λ can be interpreted as a "cycle time" (inverse of the periodic throughput). One of the purposes of this paper is to study such measures of performance. In particular, we shall see that periodicity properties of type (19) proceed from general properties of $(\max, +)$ rational series, which allow a direct computation of such cycle times.

2.5 Performance evaluation of timed automata

In the following, θ will denote a *dater* function. The three following criteria of performance are natural:

Worst case

$$\sup_{w \in \Sigma^k} (\theta | w) = (\theta | \Sigma^k) \tag{21}$$

where we have identified Σ^k to its characteristic series $\bigoplus_{w \in \Sigma^k} w$ in order to use the scalar product notation (10). $(\theta | \Sigma^k)$ represents the latest date of completion for all the sequences of k events.

Optimal case

$$\inf_{w \in \Sigma^k} (\theta | w) \tag{22}$$

with the obvious dual interpretation.

Mean case

$$\sum_{w \in \Sigma^k} (\theta | w) \times p_k(w) \tag{23}$$

where p_k is a convenient probability law on Σ^k.

As we already noticed in Example 2.4, we may consider some refinements of these measures by restricting the evaluation to a language L representing a subset of admissible events. For instance, we have the following refinement[6] of (21):

$$\sup_{w \in L \cap \Sigma^k} (\theta | w) = (\theta | L \cap \Sigma^k) \tag{24}$$

identifying as usual languages and characteristic series.

We shall see that the worst case evaluation can be obtained by simple algebraic arguments. The optimal and mean case are more difficult and much more interesting. Our approach relies on a finiteness theorem for projective semigroups of $(\max, +)$-linear maps.

[6] Indeed, all these criteria can be seen as generalizations of the conventional expectancy. Following Viot [36], we may define the "cost measure" $v : \Sigma^* \to \mathbb{R}_{\max}$, $v(w) = e$ if $w \in L$ and ε otherwise. Then, (24) rewrites $\bigoplus_{w \in \Sigma^k} (\theta | w) \otimes v(w)$ which is the counterpart of $\mathbb{E}(\theta | w)$ with respect to some probability measure (compare with (23)).

3 Worst case analysis

3.1 Main result

Let $\theta : \Sigma^* \to \mathbb{R}_{\max}$ be the dater recognized by the automaton (α, μ, β). By definition of the scalar product of series, we have, for all language L

$$(\theta|L) = \bigoplus_{w \in L} (\theta|w) \enspace .$$

Then,

$$\begin{aligned}(\theta|\Sigma^{k+1}) &= \bigoplus_{w\in\Sigma^k,\, x\in\Sigma} (\theta|wx) \\ &= \bigoplus_{w\in\Sigma^k} \alpha\mu(w)\left(\bigoplus_{x\in\Sigma}\mu(x)\right)\beta \enspace .\end{aligned} \tag{25}$$

Let

$$M = \bigoplus_{x\in\Sigma} \mu(x) \enspace .$$

It immediately follows from (25) that

$$(\theta|\Sigma^k) = \alpha M^k \beta \enspace . \tag{26}$$

It remains to analyze the asymptotic behavior of the sequence M^k. This relies on the $(\max, +)$ spectral theory (analogous to the Perron-Frobenius theory), developed in [31, 18, 9, 2, 11, 12, 28]. We first recall the definition and basic properties of the spectral radius.

3.1.1 Lemma *Let $A \in \mathbb{R}_{\max}^{n\times n}$. The following quantities are equal:*

1. $\sup\{r \in \mathbb{R}_{\max} \mid \exists u \in \mathbb{R}_{\max}^n \setminus \{\varepsilon\},\ \ Au \geq ru\}$
2. $\sup\{r \in \mathbb{R}_{\max} \mid \exists u \in \mathbb{R}_{\max}^n \setminus \{\varepsilon\},\ \ Au = ru\}$
3. $\bigoplus_{1\leq k\leq n} (\mathrm{tr} A^k)^{\frac{1}{k}} = \bigoplus_{1\leq k\leq n} \bigoplus_{i_1\dots i_k} (A_{i_1 i_2} \dots A_{i_k i_1})^{\frac{1}{k}}$
4. $\limsup_k \|A^k\|^{\frac{1}{k}}$.

This common value will be denoted by $\rho(A)$.

Of course, $a^{\frac{1}{k}}$ in the dioid algebra means $\frac{a}{k}$ in the usual algebra. In the following, it should be clear from the context whichever algebra is used. However, we shall sometimes write $a^{\otimes\frac{1}{k}}$ to avoid ambiguities. It is well known that when M is irreducible, the following cyclicity property holds [28, 2]:

$$\exists N, \exists c \geq 1,\ \forall n \geq N, \quad M^{c+n} = (\rho(M))^c M^n \tag{27}$$

where $\rho(M)$ denotes the spectral radius of M. The least value of c is called the cyclicity of M.

Recall that the representation (α, μ, β) is *trim* if

$$\forall i, j,\ \exists k, l, (\alpha M^k)_i \neq \varepsilon,\ \ (M^l \beta)_j \neq \varepsilon$$

(i.e. if each state is both accessible and co-accessible).

The following result is an easy consequence of the cyclicity property (27).

3.1.2 Proposition *(i) If M is irreducible with cyclicity c, for k large enough, we have*

$$(\theta|\Sigma^{k+c}) = \rho(M)^c(\theta|\Sigma^k) \ .$$

(ii) If (α, μ, β) is trim (but M not necessarily irreducible), we have

$$\rho(M) = \limsup_k (\theta|\Sigma^k)^{\frac{1}{k}} \ .$$

In the second case, it is of course possible to give much more precise results knowing the whole structure of M, α and β. In particular, the sequence $(\theta|\Sigma^k)$ is obtained by merging eventually geometric subsequences (cf. [15], Chapter VI) with maximal rate $\rho(M)$.

3.2 A refinement: worst case evaluation constrained in a sublanguage

We show how the worst case measure of performance restricted to an admissible sublanguage (see Formula (24)) can be computed along the same lines. We shall assume that L is a rational language. Let (α', μ', β') be a representation of L (as in §2.2,(iii)). That is, identifying L to its characteristic series, we have

$$L = \bigoplus_{w \in \Sigma^*} \alpha' \mu'(w) \beta' \, w \ .$$

The evaluation of (24) is indeed equivalent to the worst case evaluation of

$$\theta \odot l \stackrel{\text{def}}{=} \bigoplus_{w \in L} (\theta|w) \, w = \bigoplus_{w \in \Sigma^*} (\theta|w)(L|w) \, w$$

(this is the *Hadamard product* of θ and L). Since from a well known theorem of Schützenberger [5], the Hadamard product of two rational series is rational, the techniques introduced here above can be applied to (24). More precisely, $\theta \odot L$ is recognized by the "tensor product" of the linear representations of θ and L, that is, by the representation $(\alpha'', \mu'', \beta'')$ with

$$\mu''(a) = \mu(a) \otimes^t \mu'(a), \quad \alpha''_{(ij)} = \alpha_i \alpha'_j, \quad \beta''_{(ij)} = \beta_i \beta'_j$$

where $\otimes^t$ denotes the tensor product of matrices[7].

3.2.1 Proposition *Let $M' = \bigoplus_{a \in \Sigma} \mu(a) \otimes^t \mu'(a)$. Assume that M' is irreducible with cyclicity c. Then we have for k large enough:*

$$(\theta|L \cap \Sigma^{k+c}) = \rho(M')^c(\theta|L \cap \Sigma^k) \ .$$

Proof Immediate from 3.1.2,(i). ∎

[7] Recall that the tensor product of the $p \times p$-matrix A by the $q \times q$-matrix B is the $pq \times pq$-matrix

$$(A \otimes^t B)_{(ij)(kl)} = A_{ik} B_{jl} \ .$$

3.2.2 Example This allows computing the cycle times obtained by elementary means in Example 2.4. For instance, let us consider $L_2 = (a^2b)^*$. L_2 admits the following representation

$$\alpha' = \begin{bmatrix} e & \varepsilon & \varepsilon \end{bmatrix},\ \mu'(a) = \begin{bmatrix} \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & e \\ \varepsilon & \varepsilon & \varepsilon \end{bmatrix},$$

$$\mu'(b) = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ e & \varepsilon & \varepsilon \end{bmatrix},\ \beta' = \begin{bmatrix} e \\ \varepsilon \\ \varepsilon \end{bmatrix}$$

which corresponds to the automaton of Figure 5. We have:

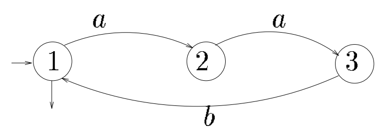

Figure 5: Automaton recognizing $(a^2b)^*$

$$M' = \bigoplus_{a\in\Sigma} \mu(a) \otimes^t \mu'(a) = \begin{bmatrix} \varepsilon & \varepsilon & 3 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 4 & 5 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 3 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 4 & 5 \\ 10 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix} .$$

According to Proposition 3.2.1, the cycle time λ is equal to $\rho(M') = \frac{16}{3}$ (this value can be obtained by applying Lemma 3.1.1,3) which accounts for Formula (20) when $l = 2$.

4 Projective finiteness of matrix semigroups

4.1 Linear projective maps

We define the matrix projective space $\mathbb{P}\mathbb{R}_{\max}^{n\times n}$ as the quotient of $\mathbb{R}_{\max}^{n\times n}$ by the parallelism relation[8]:

$$a \simeq b \Leftrightarrow \exists\lambda \in \mathbb{R}_{\max}\backslash\{\varepsilon\},\quad a = \lambda b \ . \tag{28}$$

We shall denote by $\mathfrak{p} : \mathbb{R}_{\max}^{n\times n} \to \mathbb{P}\mathbb{R}_{\max}^{n\times n}$ the canonical morphism of multiplicative semigroups. For instance

$$\mathfrak{p}\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} = \mathfrak{p}\begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} .$$

We say that a subset $S \subset \mathbb{R}_{\max}^{n\times n}$ is *projectively finite* if its projective image $\mathfrak{p}S$ is finite.

As an immediate consequence of the cyclicity result (27), we can state: $(\mathcal{P})$ *if $M \in \mathbb{R}_{\max}^{n\times n}$ is irreducible, then the semigroup generated by M, $S = \{M, M^2, M^3, \ldots\}$, is projectively finite.*

[8] By analogy with the conventional projective spaces, we should define $\mathbb{P}\mathbb{R}_{\max}^{n\times n}$ as the quotient of $\mathbb{R}_{\max}^{n\times n} \setminus \{\varepsilon\}$ by the relation $\simeq$. Here, we allow ε to belong to $\mathbb{P}\mathbb{R}_{\max}^{n\times n}$. This is a minor transgression whose interest is to make $\mathbb{P}\mathbb{R}_{\max}^{n\times n}$ become a (multiplicative) semigroup.

4.2 A first theorem

Since a rational dater writes $\theta(w) = \alpha\mu(w)\beta$, it is natural to study the image of Σ^* by μ, which is a finitely generated semigroup of matrices.

We first introduce some notation. Let $A_1, \ldots, A_p \in (\mathbb{R}_{\max})^{n\times n}$. We shall denote by $\langle A_1, \ldots, A_p\rangle$ the semigroup generated by these matrices. Let us introduce a set of p letters $\Sigma = \{a_1, \ldots, a_p\}$ and the free semigroup Σ^+ over Σ (that is, the set of non empty words over the alphabet Σ). Let $\mu : \Sigma^+ \to \mathbb{R}_{\max}^{n\times n}$ be the only morphism such that $\forall i,\ \mu(a_i) = A_i$ (i.e. $\mu(a_{i_1} \ldots a_{i_k}) = A_{i_1} \ldots A_{i_k}$). Then $\langle A_1, \ldots, A_p\rangle = \mu(\Sigma^+)$. We shall say that Σ and μ are obtained *in the canonical way* from the generators $A_1, \ldots, A_p$. We say that the semigroup $S = \langle A_1, \ldots, A_p\rangle$ is *primitive*[9] if there is an integer N such that for all word w,

$$|w| \geq N \Rightarrow \quad \forall i, j \;\; \mu(w)_{ij} > \varepsilon \ , \tag{29}$$

where $|w|$ denotes the length of the word w. When S admits a unique generator, this reduces to the notion of primitivity well known in the theory of nonnegative matrices. The following theorem extends the property §4.1,($\mathcal{P}$) to semigroups. Recall that $\mathbb{Q}_{\max} \stackrel{\text{def}}{=} (\mathbb{Q} \cup \{-\infty\}, \max, +)$ and that $\mathbb{Z}_{\max} \stackrel{\text{def}}{=} (\mathbb{Z} \cup \{-\infty\}, \max, +)$

4.2.1 Theorem *Let $A_1, \ldots, A_p \in \mathbb{Q}_{\max}^{n\times n}$. If $\langle A_1, \ldots, A_p\rangle$ is a primitive semigroup, then it is projectively finite.*

Proof Let q be the lcm of the denominators of the entries of the matrices. Since $x \mapsto x^q$ ($x^q = x \times q$ with classical notations) is an automorphism of $\mathbb{Q}_{\max}$ which maps all the entries to integers, we shall assume that $A_1, \ldots, A_p \in \mathbb{Z}_{\max}^{n\times n}$. We associate with the matrix m the following "norms":

$$\|m\| = \sup_{ij} m_{ij} \tag{30}$$

$$|m|_\wedge = \inf_{m_{ij} \neq \varepsilon} m_{ij} \tag{31}$$

(with $\inf \emptyset = +\infty$). The proof relies on the following Lemma.

4.2.2 Lemma *Let $K \in \mathbb{N}$. The set S of matrices $m \in \mathbb{Z}_{\max}^{n\times n}$ such that*

$$\frac{\|m\|}{|m|_\wedge} \leq K$$

is projectively finite.

Indeed, after normalization, we may assume that $\forall m \in S,\ \ |m|_\wedge = e$. Since there is at most $(K+2)^{n^2} - 1$ matrices $m \in \mathbb{Z}_{\max}^{n\times n}$ such that $e = |m|_\wedge$ and $\|m\| \leq K$, the Lemma is proven.

Let

$$\begin{aligned} \underline{a} &= \min(\ |A_1|_\wedge, , \ldots, |A_p|_\wedge\ ), \\ \overline{a} &= \max(\ \|A_1\|, \ldots, \|A_p\|\ ) \ . \end{aligned}$$

[9] We leave it to the reader to check that this notion is independent of the set of generators.

The primitivity assumption implies that for $w \in \Sigma^*$ long enough, we have a factorization $w = sur$ with $|s|, |r| \leq N$ and $\mu(s), \mu(r), \mu(u) > \varepsilon$ (N is the "primitivity index" satisfying (29)). Then

$$\begin{aligned} \|\mu(w)\| &= \|\mu(sur)\| \leq \|\mu(s)\| \|\mu(u)\| \|\mu(r)\| \\ &\leq (e \oplus \overline{a})^{2N} \mu(u)_{ij} \end{aligned} \tag{32}$$

for some indices ij belonging to the argmax in $\|\mu(u)\| = \sup_{ij} \mu(u)_{ij}$. Moreover

$$\mu(sur)_{kl} \geq \mu(s)_{ki} \mu(u)_{ij} \mu(r)_{jl} \geq (\underline{a} \wedge e)^{2N} \mu(u)_{ij} \ .$$

This implies that

$$\frac{\|\mu(w)\|}{|\mu(w)|_\wedge} \leq \left(\frac{e \oplus \overline{a}}{e \wedge \underline{a}} \right)^{2N} \ .$$

It remains to apply Lemma 4.2.2 to conclude. ∎

This theorem does not extend to semigroups of matrices with irrational entries. This together with some generalizations relative to the *Burnside problem* for semigroups of matrices in the $(\max, +)$-algebra is discussed in another paper [16].

4.2.3 Remark The assumptions of Theorem 4.2.1 are satisfied for a class of stochastic irreducible timed event graphs. Precisely, we consider the timed event graphs whose dater variables, $x(k)$ ($x(k)$ is a random vector with values in $\mathbb{R}^n_{\max}$) are given by equations of the form (cf. [2])

$$x(k+1) = x(k) A(k)$$

where the $A(k)$ are i.i.d. random variables taking only a finite number of values $A_1, \ldots, A_p$ (cf. §2.2,(ii)) Since dater functions are nondecreasing, we may assume that the $A(k)$ only take values greater than Id, or equivalently that $\forall i,\ A_i \geq \mathrm{Id}$. Moreover, we assume that all the A_i have the same pattern[10] which is irreducible (in other words, the durations are random but the structure of the graph is fixed and it is strongly connected). Then, it is easily checked that the semigroup $\langle A_1, \ldots, A_p \rangle$ is primitive. This indeed reduces to the following well known fact of the classical Perron-Frobenius theory: a matrix with non zero diagonal entries is irreducible iff it is primitive.

4.3 Prefix representation of projectively finite semigroups

Here, we use the term representation in the naive sense: given a projectively finite semigroup $S = \mu(\Sigma^+)$, we want to describe the (infinite) multiplication table of S in a finite convenient way.

Recall that a set P is *prefix closed* [4] if

$$uv \in P \Rightarrow u \in P \ .$$

Let $\leq$ denote the prefix order on Σ^*. This property is equivalent to P being a *lower set* (i.e. ($u \leq w$ and $w \in P$) $\Rightarrow u \in P$). A set U is a *prefix code* if

$$(uv = w \text{ and } u, w \in U) \ \Rightarrow u = w \ .$$

(a prefix code is an *antichain* for the prefix order). The prefix code associated with P is by definition $C = P\Sigma \setminus P$. For semigroups with an unique irreducible generator A, we have seen that the performance evaluation relies on the cyclicity property $A^{n+c} = \lambda^c A^n$ (the cycle time is equal to λ). So the question is: what does $A^{n+c} = \lambda^c A^n$ become for semigroups with several generators? The following proposition provides the answer. Since we may add to the semigroup S an unit, there is no loss of generality in assuming that S is a monoid.

[10] We call *pattern* of a matrix A the set of positions of the non ε entries of A

4.3.1 Proposition *Let $S = \mu(\Sigma^*)$ be a finitely generated monoid. Then the following assertions are equivalent:*

1. *S is projectively finite*
2. *there exists a finite prefix closed subset $P \subset \Sigma^*$ with associated prefix code C and two maps*

$$\varpi : C \to P, \quad \lambda : C \to \mathbb{R}_{\max}$$

   such that

$$\forall u \in C, \; \mu(u) = \lambda(u)\mu(\varpi(u)) \; . \tag{33}$$

3. *same conditions as 2, and moreover, for all $u \in C$, $\varpi(u)$ is a prefix of u.*

We think it is enough to consider the Example 4.3.2 below to understand this result, whose proof -almost trivial- is left to the reader. We just mention that P is defined as follows

$$P \overset{\text{def}}{=} \{w \in \Sigma^* \mid \forall w' < w, \quad \mathfrak{p}\mu(w) \neq \mathfrak{p}\mu(w')\} \tag{34}$$

(where $<$ denotes the strict prefix order in the case (3) of the Proposition and the strict lexicographic order in the case (2)). Hence, P is easily built by induction.

It should be noted that the condition 4.3.12 does not define an unique P. E.g, consider $\Sigma = \{a, b\}$ and assume that $\mu(a) = \mu(b) = \mu(a^2)$. Then $P = \{e, a\}$ and $P = \{e, b\}$ are valid. However, there exists a unique minimal P satisfying the stronger condition (3).

4.3.2 Example Let S be the semigroup generated by

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \; .$$

S is projectively finite. It is completely determined by the following relations:

$$AA = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$BB = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$$

$$BAB = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$$

$$AAA = e \otimes AA$$

$$AAB = 1 \otimes AA$$

$$ABA = 1 \otimes A$$

$$ABB = 2 \otimes A$$

$$BAA = 1 \otimes AA$$

$$BBA = 2 \otimes A$$

$$BBB = 2 \otimes B$$

$$BABA = 1 \otimes BA$$

$$BABB = 2 \otimes BA$$

Now take two letters a and b and $\mu : \{a,b\}^* \to S$ such that $\mu(a) = A, \mu(b) = B$. A prefix set satisfying the conditions of Proposition 4.3.1,2 is $P = \{e, a, aa, ab, ba, bab\}$. The associated prefix code is equal to

$$C = \{aaa, aab, aba, abb, baa, bba, bbb, baba, babb\} \ .$$

These two subsets of Σ^* admit the literal representation of Figure 6. P corresponds to the ends of bold arcs. The relations (33) state that any element of $\mu(C)$ is proportional to some element of $\mu(P)$. This proportionality is represented by the doted arcs on the picture. E.g., we have

$$\begin{aligned} \varpi(aaa) &= aa, \lambda(aaa) = e, \text{ and} \\ \mu(aaa) &= \lambda(aaa) \otimes \mu(\varphi(aaa)) = e \otimes \mu(aa) \ . \end{aligned}$$

These relations allow computing $\mu(w)$ without multiplying the matrices. For instance, $\mu((babb)^n) =$

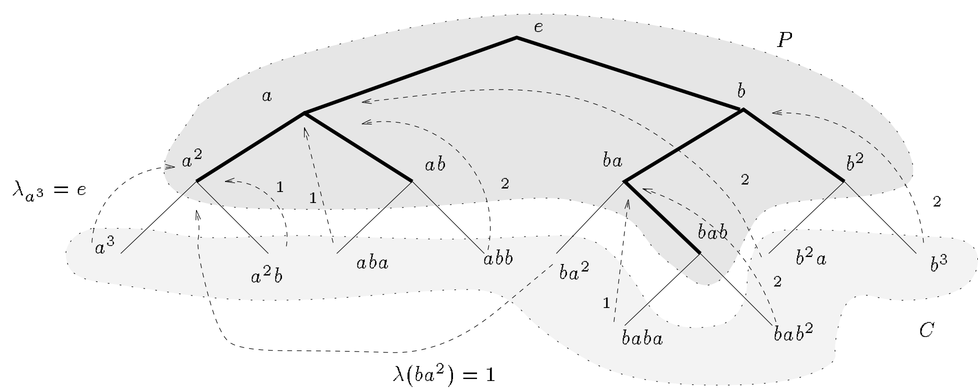

Figure 6: Literal representation of P and C

$2^n\mu((ba)^n) = 2^n 1^{n-1}\mu(ba) = 2 \times n + 1 \times (n-1) + BA = 3n - 1 + BA.$

More generally, computing $\mu(w)$ for a given word amounts to follow a path in the prefix closed set P, while performing only *scalar* multiplications. We now make this idea more precise.

Define recursively the "projection" map $\varphi : \Sigma^* \to P$ by

$$\varphi(e) = e, \ \varphi(wa) = \begin{cases} \varphi(w)a & \text{if } \varphi(w)a \in P \\ \varpi(\varphi(w)a) & \text{if } \varphi(w)a \in C. \end{cases}$$

Let $w = a_1 \dots a_n$ with $a_1, \dots, a_n \in \Sigma$. We set for $0 \le k \le n$, $w_k = a_1 \dots a_k$ and

$$m_k = \varphi(w_k) \ . \tag{35}$$

The map λ given by Proposition 4.3.1 is a priori defined only in C. We extend λ (with a slight change of notation, writing the argument as a subscript) to the whole set $P\Sigma$ by setting

$$\forall m \in P\Sigma, \quad \lambda_m = \begin{cases} e & \text{if } m \notin C \\ \lambda(m) & \text{if } m \in C. \end{cases}$$

Then, Formula (33) extends to:

$$\forall w \in P, \forall a \in \Sigma, \ \ \mu(wa) = \lambda_{wa}\mu(\varphi(wa)) \ \ .$$

The following proposition reduces the computation of $\mu(w_n)$ to the computation of $\mu(m_n)$, modulo the multiplication by some scalars.

4.3.3 Proposition *Let $w_n = a_1 \ldots a_n$, and define $m_1, \ldots, m_n$ as above. We have*

$$\mu(w_n) = \lambda_{ea_1} \otimes \lambda_{m_1a_2} \otimes \ldots \otimes \lambda_{m_{n-1}a_n} \otimes \mu(m_n) \ \ . \tag{36}$$

Therefore, $\mu(w_n)$ can be obtained with a bounded number of matrix products (corresponding to the evaluation of $\mu(m_n)$).

4.3.4 Example Let $w = abab$ and take P et C as in 4.3.2. We have

$$m_0 = e, \ m_1 = \varphi(a) = a, \ m_2 = \varphi(ab) = ab,$$
$$m_3 = \varpi(m_2a) = \varpi(a^3) = a, \ m_4 = \varphi(ab) = ab \ \ .$$

Therefore,

$$\mu(w) = \lambda_{ea}\lambda_{ab}\lambda_{aba}\lambda_{ab}\mu(m_4) = 1 \otimes AB \ \ .$$

4.3.5 Remark The $\lambda(c)$ given in Proposition 4.3.1,3 admit a simple interpretation. Since in this case, $\varpi(c)$ is a prefix of c, there exists a word u_c such that $c = \varpi(c)u_c$. Then, it easily checked that $\lambda(c) = \rho(\mu(u))$ and that

$$\mu(\varpi(c)u^n) = \rho(\mu(u))^n\mu(\varpi(c)) \ \ .$$

In other words, $(\lambda(c))^{\frac{1}{|u|}} = \rho(\mu(u))^{\frac{1}{|u|}}$ can now be interpreted as the inverse of the *periodic throughput* associated with the infinite word

$$w = \varpi(c)uuuuuu\ldots \tag{37}$$

That is, if w_n denotes the word composed of the n first letters of w, we have $\mu(w_{n+|u|}) = \lambda(c)\mu(w_n)$ for n large enough. More generally, for any sequence of words $w_n \in \Sigma^n, w_n \leq w_{n+1}$, it is easily checked that for all i, j such that $\mu(w_n)_{ij}$ does not takes the value ε when $n \to \infty$,

$$\begin{aligned} \inf_{c\in C} \lambda(c)^{\frac{1}{|u_c|}} &\leq \liminf_n (\mu(w_n)_{ij})^{\frac{1}{n}} \\ \limsup_n (\mu(w_n)_{ij})^{\frac{1}{n}} &\leq \sup_{c \in C} \lambda(c)^{\frac{1}{|u_c|}} \ \ . \end{aligned}$$

Of course, the bounds are attained for words of the form (37).

5 Application to performance evaluation

5.1 Kolmogorov equation of stochastic timed automata

Let p_k denote the probability on Σ^k such that $\forall w_k \in \Sigma^k$ (we write $w_k = a_1 \dots a_k$ with $\forall i, a_i \in \Sigma$),

$$p_k(w_k) = p_k(a_1 \dots a_k) = p(a_1) \times \dots \times p(a_k) \tag{38}$$

where $p(a_i) \geq 0, \sum_i p(a_i) = 1$. We consider

$$\rho_{ij}^{(k)} \stackrel{\text{def}}{=} \mathbb{E}\mu(w_k)_{ij} = \sum_{w \in \Sigma^k} \mu(w)_{ij} p_k(w) \ .$$

In order to understand which behavior of $\rho_{ij}^{(k)}$ we may expect as $k \to \infty$, we first consider the simple case of a semigroup with a unique generator $A = \mu(a_1)$ (deterministic case, $p(a_1) = 1$). It follows from the cyclicity theorem (27) that the following limit exists

$$\ell_{ij} = \lim_k (\rho_{ij}^{(k)})^{\otimes \frac{1}{k}} = \lim_k \frac{1}{k} \times \rho_{ij}^{(k)} \tag{39}$$

with $\ell_{ij} = \rho(A)$ (spectral radius), independent of ij. If A is not positive but irreducible, some little care is needed, for $\rho_{ij}^{(k)}$ can take the ε value. Hence, we set

$$\ell_{ij} \stackrel{\text{def}}{=} \lim_{k \neq \varepsilon,\ k \to \infty} (\rho_{ij}^{(k)})^{\otimes \frac{1}{k}} \ . \tag{40}$$

Of course, we have to specify for which values of k $\rho_{ij}^{(k)} = \varepsilon$. Let us recall that a subset L of $\mathbb{N}$ is *rational* iff it is equal to the union of a finite set and a finite number of arithmetic progressions ([13], Chapter V, Proposition 1.1), or equivalently, if it is ultimately invariant under some translation, i.e. iff there exists $N \in \mathbb{N}$ and $c \geq 1$ such that

$$\forall k \geq N, \quad (k \in L \iff k + c \in L) \ .$$

It clearly follows from $A^{n+c} = \rho(A)^c A^n$ for n large enough that

$$K_{ij} \stackrel{\text{def}}{=} \{k \in \mathbb{N} \mid \ \rho_{ij}^{(k)} = \varepsilon\} \ \text{ is a rational subset of } \mathbb{N}, \tag{41}$$

and this rational subset is easily computable. Thus, the asymptotic behavior of $\rho_{ij}^{(k)}$ is properly determined by (40) together with (41). We next generalize these two properties to semigroups with several generators. The ℓ_{ij} which measure some "mean asymptotic performance" are particular cases of the "Lyapunov exponents" studied by Baccelli, so called by analogy with the conventional algebra. Under some "irreducibility" assumptions[11], the existence of ℓ_{ij} can be obtained by subadditive ergodic arguments [1, 2], as in the case of products of random matrices in the usual algebra [7]. Indeed, it is shown that $\lim_k (\mu(w_k)_{ij})^{\frac{1}{k}} = \ell_{ij}$ a.s. and that ℓ_{ij} does not depend of ij (but of course, $\rho_{ij}^{(k)}$ does). The exact value of ℓ_{ij} is not known in general. Some bounds are given in [3, 2]. Olsder has shown that in certain cases, a finite Markov chain can be written. Olsder approach is equivalent to saying that the random vectors $x(0), x(1), \dots$ defined by

$$x(0) = x_0, \quad x(n) = x(n-1)A(n)$$

[11] Here, the simplest version of these conditions requires that all the matrices $\mu(a_i)$ have non ε entries. Of course, much more precise conditions might be given.

-where the $A(i)$ are some specific i.i.d. random matrices- only take a finite number of values in the projective space. Our approach is an extension of Olsder's technique. The main novelties by comparison with Olsder is that: (i) the probabilistic problem (compute ℓ_{ij}) is now reduced to an algebraic one (decide if a semigroup of projective linear maps is finite); (ii) the exact value at the k-th step $\rho_{ij}^{(k)}$ is obtained; (iii) it is not assumed that the limit $x(n) - x(n-1)$ exists.

We next show that the mean case performance $\rho^{(k)}$ and its asymptotic mean ℓ can be exactly computed from the Kolmogorov equation of an associated Markov chain over the prefix closed subset P. The idea of writing a Kolmogorov equation originated from a discussion with Jean-Pierre Quadrat, to whom the author is indebted.

Induced Markov Chain We assume that $\mu(\Sigma^*)$ is projectively finite. Thus, we have a finite prefix closed subset $P \subset \Sigma^*$ with associated prefix code C and λ, ϖ maps as in (33),

We consider a sequence $\{w_k\}$ of random variables with values in Σ^* such that $w_k = a_1 \ldots a_k$ where the a_i are i.i.d. random variables with values in Σ and probability law p (the same as in (38)). $\{\mu(w_k)\}_k$ is thus a "matrix random walk". Let $m_k = \varphi(w_k)$. It is clear that $\{m_k\}_k$ is a Markov chain with states P and transition matrix:

$$\mathcal{M} : \ \mathcal{M}_{mm'} = \sum_{a \in \Sigma,\, \varphi(ma) = m'} p(a)$$

with the convention $\sum_{a \in \emptyset} p(a) = 0$. This Markov chain can easily be visualized on Figure 6 by identifying the elements of C with some elements of P according to the backward arrows.

We think it is not superfluous to rewrite now Formula (36) with standard notation:

$$\mu(w_n) = \lambda_{ea_1} + \lambda_{m_1 a_2} + \ldots + \lambda_{m_{n-1} a_n} + \mu(m_n) \tag{42}$$

where for a scalar s and a matrix m, $s + m$ denotes the matrix:

$$(s + m)_{ij} \stackrel{\text{def}}{=} s + m_{ij} \ . \tag{43}$$

From (42), we see that $\mathbb{E}(\mu(w_k))$ is the sum of two quantities.

(i) $\mathbb{E}(\mu(m_k))$, which is bounded (for $m_k \in P$)

(ii) A sum of transition costs of the Markov chain.

Let us introduce

$$v^k(x) = \mathbb{E}[\lambda_{m_0 a_1} + \lambda_{m_1 a_2} + \ldots + \lambda_{m_{k-1} a_k} | m_0 = x] \ .$$

Then, we get from (42)

$$\mathbb{E}(\mu(w_k)) = \mathbb{E}(\mu(m_k)) + v^k(e) \ . \tag{44}$$

It remains to define

$$c(m) = \sum_{a \in \Sigma} \lambda_{ma} p(a) \ .$$

5.1.1 Theorem *If $\mu(\Sigma^*)$ is a projectively finite semigroup, then we have with the above notation*

$$\rho^{(k)} = \mathbb{E}(\mu(w_k)) = v^k(e) + \sum_{m \in P} \left(\mathcal{M}^k\right)_{e,m} \mu(m) \tag{45}$$

where v^k is given by the Kolmogorov equation

$$v^k = c + \mathcal{M} v^{k-1}, v^0 = 0 \ .$$

The Kolmogorov equation implies that

$$v^k = (\mathrm{Id} + \ldots + \mathcal{M}^{k-1})c \ . \tag{46}$$

Let $\mathcal{P}$ be the spectral projector of $\mathcal{M}$ for the eigenvalue 1. The ergodic theorem for Markov chains implies that

$$\lim_k \frac{v^k}{k} = \mathcal{P}c \ . \tag{47}$$

Let

$$K_{ij} = \{k \in \mathbb{N} \mid \exists m \in P,\ \mathcal{M}^k_{em} \neq 0 \ \text{ and } \mu(m)_{ij} = \varepsilon\}$$

Since the sum at the right hand of (45)

$$\sum_{m \in P} \left(\mathcal{M}^k\right)_{e,m} \mu(m)_{ij}$$

is obviously bounded for $k \in \mathbb{N} \setminus K_{ij}$, we have

$$\lim_{k \notin K_{ij},\, k \to \infty} \frac{\rho^{(k)}_{ij}}{k} = (\mathcal{P}c)_e \ .$$

Moreover, it is standard Perron-Frobenius theory that for all $m \in \mathcal{P}$,

$$L_m \stackrel{\text{def}}{=} \{k \in \mathbb{N} \mid \ \mathcal{M}^k_{em} = 0\}$$

is a rational subset[12] of $\mathbb{N}$, hence $K_{ij} = \cup_{m \in P, \mu(m)_{ij} = \varepsilon} L_m$ is rational (finite union of rational subsets). Since the two properties (40) and (41) are satisfied, we have characterized the asymptotic mean case performance. In summary:

5.1.2 Corollary *Under the same assumption as in 5.1.1, the "Lyapunov exponent" $\mathfrak{l}_{ij}$ defined by (40) exists and is equal to*

$$\mathfrak{l}_{ij} = (\mathcal{P}c)_e \ .$$

Thus, the projective finiteness condition implies that the Lyapunov exponent is independent of ij. This could also be checked by elementary means (if two Lyapunov exponents were distinct, then the "projective width" of the semigroup would be infinite).

5.1.3 Example We take again the semigroup S of Example 4.3.2, with probabilities $p(a) = u, p(b) = v$ (such that $u + v = 1$). We obtain the following Markov matrix

$$\mathcal{M} = \begin{array}{c} \\ e \\ a \\ b \\ a^2 \\ ab \\ ba \\ b^2 \\ bab \end{array} \begin{array}{c} \begin{array}{cccccccc} e & a & b & a^2 & ab & ba & b^2 & bab \end{array} \\ \begin{bmatrix} 0 & u & v & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u & v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u & v & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u & 0 & 0 & 0 & v \\ 0 & u & v & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array} \tag{48}$$

[12] The proof is as follows. Obviously, the dioid of subsets of $\mathbb{N}$ is isomorphic to $\mathbb{B}[[X]]$ (dioid of formal series in a single indeterminate with boolean coefficients), the isomorphism being the map $L \mapsto \bigoplus_{k \in L} X^k$. Consider the boolean matrix $M : M_{p,q} = e$ if $\mathcal{M}_{p,q} \neq 0$ and $M_{p,q} = \varepsilon$ otherwise. Set $\alpha_p = \delta_{e,p}$ (Kronecker's δ, i.e. $\delta_{e,p} \stackrel{\text{def}}{=} e$ if $p = e$, $\delta_{e,p} = \varepsilon$ otherwise) and $\beta_q = \delta_{qm}$. Then $\complement L_m = \{k \mid \ M^k_{em} \neq \varepsilon\}$ is clearly recognized by the linear representation (α, M, β). Hence, its complementary L_m is also rational.

with

$$c = \begin{array}{c} \begin{matrix} e & a & b & a^2 & ab & ba & b^2 & bab \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & v & u+2v & u & 2 & u+2v \end{bmatrix} \end{array}$$

(for instance, the value c_{a^2} is obtained as $c_{a^2} = u \times 0 + 1 \times v = v$). The unique invariant measure is

$$z:\ z_m = \begin{cases} 1 & \text{si } m = a^2 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the spectral projector is $\mathcal{P} = \mathbf{1}z$, where $\mathbf{1}$ denotes the constant vector with entries 1. Finally, the $(\max, +)$ Lyapunov exponent is equal to

$$\mathfrak{l} = (\mathcal{P}c)_e = zc = c_{a^2} = v \ .$$

5.1.4 Remark It is sometimes possible to compute the Lyapunov exponent in the case of a non primitive semigroup of matrices. Consider the factorization

$$\Sigma^* = (e \oplus \ldots \oplus \Sigma^{c-1})(\Sigma^c)^* \ .$$

Then, we claim that the computation of the Lyapunov exponent(s) of $\mu : \Sigma^* \to \mathbb{R}_{\max}^{n\times n}$ reduces to the computation of the Lyapunov exponents of the induced morphism $\mu^{(c)} : (\Sigma^c)^* \to \mathbb{R}_{\max}^{n\times n}$ (such that $\mu^{(c)}(w) = \mu(w)$ for all $w \in (\Sigma^c)^*$). We shall not discuss this here in full generality, but we just give a sketch of procedure for computing these exponents under some coarse assumptions and illustrate it with a simple example. 1/ *We assume that the matrix* $M = \bigoplus_{a\in\Sigma} \mu(a)$ *is irreducible.* Let c be the cyclicity of M. Then by a well known fact of the Perron-Frobenius theory, M^c is bloc diagonal. Hence, the morphism μ^c is the direct product of some morphisms, that is, there exists a partition $n = n_1 + \ldots + n_c$ and c morphisms: $\mu_i : (\Sigma^c)^* \to \mathbb{R}_{\max}^{n_i\times n_i}$ $(1 \le i \le c)$ such that $\mu^{(c)}(w) = \mathrm{diag}(\mu_1(w), \ldots, \mu_c(w))$. 2/ *We shall assume that all the morphisms* μ_i *have primitive images* (we always assume that the entries of $\mu_i(w)$ are rational). Then, the Lyapunov exponent associated with each μ_i can be computed by Theorem 5.1.2. For instance, consider the matrices

$$A = \begin{bmatrix} \varepsilon & e \\ e & \varepsilon \end{bmatrix}, \quad B = \begin{bmatrix} \varepsilon & e \\ 1 & \varepsilon \end{bmatrix} \ .$$

with probabilities $p(A) = p, p(B) = q$. We are reduced to computing the Lyapunov exponents of the following morphism:

$$\mu^{(2)} : \{aa, ab, ba, bb\}^* \to \mathbb{R}_{\max}^{2\times 2}$$

with probabilities $p(aa) = p^2, p(ab) = p(ba) = pq, p(b^2) = q^2$. We have

$$\mu(aa) = \mathrm{Id},\ \mu(ab) = \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & e \end{bmatrix},$$

$$\mu(ba) = \begin{bmatrix} e & \varepsilon \\ \varepsilon & 1 \end{bmatrix}, \quad \mu(b^2) = 1\mathrm{Id} \ .$$

The Lyapunov exponent of the first diagonal bloc is

$$\mathfrak{l}_1 = p^2 \times 0 + 1 \times pq + 0 \times pq + 1 \times q^2 = q \ .$$

The Lyapunov exponent of the second diagonal bloc $\mathfrak{l}_2$ is computed in an analogous way and is equal to $\mathfrak{l}_1$. Hence, the "maximal lyapunov exponent" defined by

$$\mathfrak{l} \stackrel{\text{def}}{=} \lim_k \mathbb{E}\|\mu(w)\|^{\frac{1}{k}} \tag{49}$$

is well defined and equal to $(\mathfrak{l}_1)^{\frac{1}{2}} = 1/2 \times q$ (there is a factor of normalization $1/2$ due to the fact that words of length k in the alphabet $\{aa, ab, ba, bb\}$ are of length $2k$ in the alphabet $\Sigma = \{a, b\}$).

5.1.5 Remark We have adopted here a presentation which focuses on the induced semigroup in the projective space: hence, the "prefix representation" does not depend on the initial condition α. It should be noted that if a specific value of α is given, it is less expensive to compute the finite set $\mathfrak{p}\alpha\mathfrak{p}\mu(\Sigma^*)$ (the finite orbit of $\mathfrak{p}\alpha$ under the action of the semigroup $\mathfrak{p}\mu(\Sigma^*)$). In particular, since the projective finiteness condition implies that $\mathfrak{l} = \lim_k(\mathbb{E}\|\alpha\mu(w_k)\|)^{\frac{1}{k}}$ does not depend of the initial condition α (as soon as $\alpha_i \neq \varepsilon, \forall i$), if we only consider the computation of $\mathfrak{l}$, it is enough to compute the finite set $\mathfrak{p}\alpha\mathfrak{p}\mu(\Sigma^*)$ for such an arbitrary α. For instance, with $\alpha = [e, e]$, we obtain the finite set shown on Figure 7. We have written on the picture $\alpha\mu(b) \stackrel{1a}{\to} \alpha$ on the picture to express that

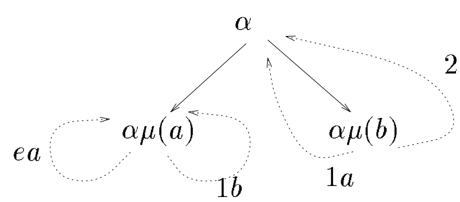

Figure 7: Orbit of $\alpha = [e, e]$

$\alpha B \otimes A = 1 \otimes \alpha$. This implies that the asymptotic mean case analysis of $\alpha\mu(w)$ can be performed by reasoning on a Markov chain similar to (48) but with only with 3 states. In this case, we are reduced to the original approach of Olsder (see [2], Chapter 9).

5.1.6 (Extension to Markovian probabilities) For a projectively finite semigroup $S = \mu(\Sigma^+)$, it is also possible to compute the Lyapunov exponent when the probability measure p is Markovian [20], that is

$$\begin{cases} p(a_1 \dots a_k) = p(a_1)P(a_1, a_2) \dots P(a_{k-1}a_k) \\ p(a) = \sum_{b \in \Sigma} \alpha(b)M(b, a) \end{cases} \tag{50}$$

for some stochastic vector α (i.e. $\sum_a \alpha(a) = 1$) and Markov matrix P. In other words, we consider a Markov chain in the free monoid, $w_1, w_2 \dots$, (with $w_i \in \Sigma^i$) such that $p(w_{i+1} = a_1 \dots a_{i+1} | w_i = a_1 \dots a_i) = P(a_i, a_{i+1})$. Let us define the map $k : \Sigma^* \to \Sigma$: $k(a_1 \dots a_i) = a_i$ (k retains only the last letter of the word). Then, it is easily seen that $(\mathfrak{p}\mu(w_i), k(w_i))_{i=1,2\dots}$ is a Markov chain with finite state, for which the above discussion can be easily extended.

5.2 Hamilton-Jacobi equation for the optimal behavior

The word $w \in \Sigma^n$ is now seen as a control. Let us introduce the minimal completion time of the n first events:

$$\inf_{w \in \Sigma^n} \theta(w) \ .$$

Since $\theta(w) = \alpha\mu(w)\beta$ and α, β are constants, we shall only consider

$$r_{ij}^{(n)} = \inf_{w \in \Sigma^n} \mu(w)_{ij} \tag{51}$$

together with the associated mean optimal performance

$$r_{ij} = \lim_{n \to \infty,\ r_{ij \neq \varepsilon}^{(n)}} \frac{r_{ij}^{(n)}}{n} \ .$$

Induced decision process Let $w = a_1 \dots a_n$ be an optimal word (such that the minimum is attained in (51)), and let us introduce the prefixes $w_0, w_1, \dots w_n$ and their projections $m_0, m_1, \dots, m_k$ as in (35). We pass from w_k to w_{k+1} by choosing a letter in an optimal way. Equivalently, we pass from m_k to m_{k+1}. This "decision process" with finite state (since $m_k \in P$) appears as the $(\min, +)$ analogous of a finite Markov chain (this is a particular case of Markov chains over the $(\min, +)$ algebra introduced by Viot [36]). Looking at (42), minimizing $\theta(w)$ amounts to minimizing a sum of transition costs of the "decision process" plus a final cost. Therefore, we introduce

$$v^k(x) = \inf_{a_1,\dots,a_k,m_k=x} \lambda_{a_1} + \lambda_{m_1 a_2} \dots + \lambda_{m_{k-1} a_k} \ .$$

where for each subsequence of l letters $a_1, \dots, a_l$, we have set $m_l = \varphi(a_1 \dots a_l)$. v^k satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$v^k(x) = \inf_{y \in P,\ a \in \Sigma, \varphi(ya)=x} [v^{k-1}(y) + \lambda_{\varphi(y)a}], \tag{52}$$

$$v^0(x) = \begin{cases} 0 & \text{if } x = e \\ +\infty & \text{otherwise} \end{cases} .$$

There is a simpler matrix expression for (52). Introduce the dioid $\overline{\mathbb{R}}_{\mathbf{min}} \stackrel{\text{def}}{=} (\mathbb{R} \cup \{-\infty, +\infty\}, \min, +)$ (where addition and product are denoted by $\oplus'$ and $\otimes'$ together with the convention $(-\infty) \otimes' (+\infty) = +\infty$) and define the matrix $\Lambda \in \mathbb{R}_{\mathbf{min}}^{P \times P}$ by

$$\Lambda_{mm'} = \inf_{a \in \Sigma, \varphi(ma)=m'} \lambda_{\varphi(m)a} = \bigoplus_{a \in \Sigma, \varphi(ma)=m'}^{\prime} \lambda_{\varphi(m)a}$$

(with the convention $\inf \emptyset = +\infty$). Then, (52) rewrites linearly

$$v^k = v^{k-1} \otimes' \Lambda \ .$$

This linear HJB equation plays a role analogous to the Kolmogorov equation for the above introduced Markov chain.

5.2.1 Theorem *If $\mu(\Sigma^*)$ is projectively finite, then we have with the above notation*

$$r^{(k)} = \inf_{m \in P} [v^k(m) + \mu(m)] = \bigoplus_{m \in P}^{\prime} \left(\Lambda^{\otimes' k}\right)_{e,m} \otimes' \mu(m) \ .$$

We shall denote by $\rho_{\min,+}(\Lambda)$ the spectral radius of Λ in the $(\min, +)$ algebra (that is the dual of the spectral radius given in 3.1.1). We obtain as an immediate corollary of Theorem 5.2.1:

5.2.2 Corollary *Under the same assumption as in 5.2.1, we have*

$$r_{ij} = \lim_{k,\ r_{ij}^{(k)} \neq \varepsilon} (r_{ij}^{(k)})^{\otimes \frac{1}{k}} = \lim_{k,\ r_{ij}^{(k)} \neq \varepsilon} \frac{r_{ij}^k}{k} = \rho_{\min,+}(\Lambda) \ .$$

Therefore, the projective finiteness condition implies that the limit r_{ij} is independent of the indices ij. Of course, the set $\{k \in \mathbb{N} \mid r_{ij}^{(k)} = \varepsilon\}$ is an easily computable rational subset of $\mathbb{N}$, by an argument analogous to that given for the mean case analysis.

5.2.3 Example For the semigroup of the example 4.3.2, we have

$$\Lambda = \begin{array}{c} \\ e \\ a \\ b \\ a^2 \\ ab \\ ba \\ b^2 \\ bab \end{array} \begin{array}{c} \begin{array}{cccccccc} e & a & b & a^2 & ab & ba & b^2 & bab \end{array} \\ \begin{bmatrix} \varepsilon & e & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e & e & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & e & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon & e \\ \varepsilon & 2 & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon \end{bmatrix} \end{array}$$

$\rho(\Lambda) = e$ (indeed, $a^2 \to a^2$ is the unique "critical circuit"[13] , i.e. $\Lambda_{a^2a^2} = (\Lambda_{a^2a^2})^{\frac{1}{1}}$ is the unique term attaining the bound in 3.1.1,(3). For $i = j = 1$, the unique optimal word of length k is $w = a^k$ for which $\mu(w)_{ij} = e$.

It should be noted that an analogous "Markovian" optimization problem has previously been solved by Olsder in a more particular setting (see [2], Chapter 9).

5.2.4 Remark Assume for simplicity that $r_{ij}^{(n)}$ does not takes ε values for large n. Then, from the projective finiteness assumption, we obtain the "linear" asymptotic behavior $r_{ij}^{(n)} \simeq C \times n$ with $C = \rho_{\min,+}(\Lambda)$. Without this projective finiteness assumption, the behavior of the "optimal" performance $r^{(n)}$ can be very different. Simon [33, 34, 35] has exhibited a family of automata for which $r_{ij}^{(n)}$ is of the order $\sqrt[p]{n}$ (p can be an arbitary integer).

6 Approximation of the Lyapunov exponent

6.1 About Baccelli's lower bound of the Lyapunov exponent

It has been noted that even when dealing with a projectively finite semigroup of matrices, the size of this semigroup may be too important to allow the exact computation of the Lyapunov exponent. An alternate approach consists in giving simple computable bounds for this exponent. We refer the reader to Bacelli [1] [2]. Baccelli's lower bound is based on stochastic ordering result [2]. Here, our purpose is to give another proof of Baccelli's bound, purely algebraic.

Here above, we have shown deduced the existence of the Lyapunov exponents from projective finiteness properties of semigroups. More generally, we can always define the *maximal* Lyapunov exponent of a product of i.i.d. random matrices in $\mathbb{R}_{\max}^{n\times n}$ by definition

$$\iota \stackrel{\text{def}}{=} \lim_k \mathbb{E}\|A(1)\dots A(k)\|^{\frac{1}{k}} = \inf_k \mathbb{E}\|A(1)\dots A(k)\|^{\frac{1}{k}} \ . \tag{53}$$

This quantity is well defined (perhaps equal to $\varepsilon = -\infty$) as soon as $\forall i, j, A_{ij} \oplus e$ is integrable. The fact the limit exists and is equal to the infimum results from standard subadditive arguments [23, 2]. Baccelli obtains the following lower bound.

$$\rho(\mathbb{E}[A]) \le \iota \ . \tag{54}$$

[13] See [9, 2] for the graphical interpretation of ρ. The reader interested by the effective computation of the spectral radius is referred to Karp [22]

Here, we only consider the case where $A(i)$ belongs to a finite set, and give an algebraic proof based on Hadamard products of matrices in the $(\max,+)$-algebra.

We shall use the notation of §5.1: $w_k = a_1 \dots a_k$ denotes a random word on Σ^k with probability law $p(w_k) = p(a_1) \dots p(a_k)$.

6.1.1 Proposition *We have for all $k \geq 1$,*

$$\rho(\mathbb{E}(\mu(w_1))) \leq (\rho(\mathbb{E}\mu(w_k))^{\frac{1}{k}} \leq \iota \ . \tag{55}$$

Moreover, if the projective width $\Delta(\mu)$ is finite, then

$$\lim_k (\rho(\mathbb{E}\mu(w_k))^{\frac{1}{k}} = \iota \ .$$

Of course, the interest of $(\rho(\mathbb{E}\mu(w_k))^{\frac{1}{k}}$ by comparison with $\rho(\mathbb{E}(\mu(w_1))$ is only theoretical since the complexity of computing $\mathbb{E}(\mu(w_k)^{\frac{1}{k}})$ is exponential in k. We shall prove this result in the remaining part of this section. Before, we think it is worthwile making a short digression towards the usual theory of products of random matrices. The reader will certainly have noticed that the blueprint of this paper consists in translating some classical results of the theory of nonnegative matrices to the $(\max,+)$-case. Conversely, we may wonder if some $(\max,+)$ theorems furnish some new results when translated to the usual algebra. Here is a modest example a such a transfer: Baccelli's bound (54) also holds in the conventional algebra, but with the reverse inequality !

6.1.2 Proposition *Let μ be a morphism $\Sigma^* \to (\mathbb{R}^+)^{n\times n}$ (equipped with the conventional matrix product), and let $\iota = \lim_k \mathbb{E}\|\mu(w_k)\|^{\frac{1}{k}}$. Then, we have*[14]

$$\iota \leq \rho(\mathbb{E}(\mu(w_1))) = \rho(\sum_{a\in\Sigma} \mu(a)p(a)) \ .$$

This is nothing but the observation that the exact evaluation of the worst case performance in the $(\max,+)$-case yield *inequalities* for the probabilistic case. Therefore, we just mimic the proof of Proposition 3.1.2.

Proof of Proposition 6.1.2. We have

$$\begin{aligned}
\mathbb{E}(\mu(w_k)_{ij}^{\frac{1}{k}}) &\leq (\mathbb{E}[\mu(w_k)_{ij}])^{\frac{1}{k}} \quad \text{(Jensen inequality)} \\
&= \left(\sum_{w\in\Sigma^k} \mu(w)_{ij} p(w)\right)^{\frac{1}{k}} \\
&= \left[\left((\sum_{a\in\Sigma} \mu(a)p(a))^k\right)_{ij}\right]^{\frac{1}{k}} \qquad (56)\\
&\quad \text{(morphism property of } \mu \text{ and } p).
\end{aligned}$$

Let $M = \sum_{a\in\Sigma} \mu(a)p(a)$. The property follows from $\limsup_k (M_{ij}^k)^{\frac{1}{k}} \leq \rho(M)$. ■

[14] here, $\rho(B)$ denotes the conventional Perron-Frobenius eigenvalue of B.

Indeed, in the usual algebra (but unfortunately, not in the dioid's case), the scope of bounds of type 6.1.2 is not limited to the i.i.d. case. A similar bound also holds for a rational probability measure [20] on the free monoid. We assume that p admits a linear representation (α', μ', β'):

$$p(w) = \alpha' \mu'(w) \beta' \tag{57}$$

($\mu' : \Sigma^* \to (\mathbb{R}^+)^{p\times p}$ for some p, $\alpha' \in (\mathbb{R}^+)^{1\times p} \beta' \in (\mathbb{R}^+)^{p\times 1}$, for $p = 1$, we recover the i.i.d case).

6.1.3 Proposition *Let μ be a morphism $\Sigma^* \to (\mathbb{R}^+)^{n\times n}$ and p be the rational probability measure with representation (57). Then*

$$\mathfrak{l} \leq \rho(\sum_{a\in\Sigma} \mu(a) \otimes^t \mu'(a)) \tag{58}$$

where $\otimes^t$ denotes the tensor product.

The bound (58) admits a particularly simple form for Markovian probability measures. Assume that $p(a_1 \dots a_k) = p(a_1)P(a_1, a_2) \dots P(a_{k-1}, a_k)$ and that $p(a) = \sum_{b\in\Sigma} \alpha'(b) P(b, a)$ for some probability vector α' and Markov matrix P. Then, the morphism μ' is as follows

$$\mu' : \Sigma^* \to (\mathbb{R}^+)^{\Sigma\times\Sigma},\ \mu'(a)_{bc} = \begin{cases} P(b,c) & \text{if } c = a \\ 0 & \text{otherwise} \end{cases}$$

(take $\beta'_a = 1$, $\forall a \in \Sigma$ to obtain a linear representation (57)). Then, the matrix $M = \sum_{a\in\Sigma} \mu(a) \otimes^t \mu'(a)$ is given in bloc form by

$$M_{ab} = P(a, b)\mu(b) \ .$$

For instance, in the case of an alphabet reduced to two letters, we obtain

$$\mathfrak{l} \leq \rho \begin{bmatrix} P(a,a)\mu(a) & P(a,b)\mu(b) \\ P(b,a)\mu(a) & P(b,b)\mu(b) \end{bmatrix}$$

Proof of 6.1.3: We have

$$\begin{aligned} \sum_{w\in\Sigma^k} \mu(w)_{ij} p(w) &= \sum_{w\in\Sigma^k} \sum_{ls} \alpha'_l (\mu(w) \otimes^t \mu'(w))_{il,js} \beta'_s \\ &= \sum_{ls} \alpha'_l \left((\sum_{a\in\Sigma} \mu(a) \otimes^t \mu'(a))^k \right)_{il,js} \beta'_s \ . \end{aligned}$$

We conclude by arguing as in the proof of 6.1.2. ∎

6.2 Preparation for Proposition 6.1.1

We first give some inequalities related with $(\max, +)$ spectral radii and Hadamard products.

The *Hadamard product* (or Schur product) $A \odot B$ is by definition the componentwise product of A and B, that is

$$(A \odot B)_{ij} = A_{ij} B_{ij} \ .$$

Consequently, for $x \in \mathbb{R}$, we shall denote by $A^{\odot x}$ the x-th Hadamard power of A, that is

$$(A^{\odot x})_{ij} = (A_{ij})^x = A_{ij} \times x \ .$$

6.2.1 Lemma *For all $A, B, C, D \in \mathbb{R}_{\max}^{n\times n}$ and $x \in \mathbb{R}$, we have*

1. $(A \odot B) \otimes (C \odot D) \leq (A \otimes C) \odot (B \otimes D)$
2. $\rho(A \odot B) \leq \rho(A)\rho(B)$
3. $\rho(A^{\odot x}) = (\rho(A))^x$.

Proof (1) We have

$$\begin{aligned}
[(A \odot B) \otimes (C \odot D)]_{ij} &= \bigoplus_k A_{ik}B_{ik}C_{ik}D_{kj} \\
&\leq \left(\bigoplus_k A_{ik}B_{kj}\right)\left(\bigoplus_l C_{il}D_{lj}\right) \\
&= [(A \otimes C) \odot (B \otimes D)]_{ij} \ .
\end{aligned}$$

(2): From (1), we have for all $k \geq 1$, $\mathrm{tr}((A \odot B)^k)^{\frac{1}{k}} \leq \mathrm{tr}(A^k \odot B^k)^{\frac{1}{k}} \leq (\mathrm{tr}(A^k))^{\frac{1}{k}}(\mathrm{tr}(B^k))^{\frac{1}{k}} \leq \rho(A)\rho(B)$. Applying Lemma 3.1.1,3, we are done. Another interesting proof based on tensor products is given in note[15].

We shall also need the following Lemma, which is almost obvious.

6.2.2 Lemma *For all $A \in \mathbb{R}_{\max}^{n\times n}$ and $k \geq 1$:*

$$\rho(A^k) = (\rho(A))^k \quad . \tag{59}$$

Proof $\rho(A^k) \leq (\rho(A))^k$ follows immediately from Lemma 3.1.1,3 (this can be rephrased in terms of graphs, any circuit of length l of A^k can be identified with a circuit of length kl of A). The converse inequality follows from 3.1.1,2. For if $Au = ru$, then $A^k u = r^k u$, hence $\rho(A^k) \geq (\rho(A))^k$. ■

6.3 Proof of the Proposition

6.3.1 Lemma *We have*

$$\mathbb{E}[\mu(w_{k+l})] \geq \mathbb{E}[\mu(w_k)] \otimes \mathbb{E}[\mu(w_l)] \quad . \tag{60}$$

Proof The expectancy writes in $(\max, +)$ notations:

$$\begin{aligned}
\mathbb{E}[\mu(w_{k+l})] &= \prod_{w\in\Sigma^{k+l}}^{\odot} \mu(w)^{\odot p(w)} \\
&= \prod_{u\in\Sigma^k, v\in\Sigma^l}^{\odot} \mu(u)^{\odot p(uv)}\mu(v)^{\odot p(uv)} \\
&\geq \left(\prod_{(u,v)\in\Sigma^k\times\Sigma^l}^{\odot} \mu(u)^{\odot p(uv)}\right) \otimes
\end{aligned}$$

[15] Recall that the tensor product $A \otimes^t B$ of two $n \times n$-matrices is the $n^2 \times n^2$-matrix $(A \otimes^t B)_{(ij)(kl)} = A_{ik}B_{jl}$. We claim that $\rho(A \otimes^t B) = \rho(A)\rho(B)$. Indeed, since an arbitrary circuit of $A \otimes^t B$ writes $A_{i_1 i_2}B_{j_1 j_2} \dots A_{i_k i_1}B_{j_k j_1} = (A_{i_1 i_2} \dots A_{i_k i_1})(B_{j_1 j_2} \dots B_{j_k j_1})$, we have by 3.1.1,3 that $\rho(A \otimes^t B) \leq \rho(A)\rho(B)$. The other inequality results from the fact that $Au = ru$ and $Bv = sv \Rightarrow (A \otimes^t B)(u \otimes v) = (Au) \otimes^t (Bv) = rsu \otimes^t v$ which is a general property of tensor products. Since $A \odot B$ is a principal submatrix of $A \otimes^t B$, we have $\rho(A \odot B) \leq \rho(A \otimes^t B) = \rho(A) \otimes \rho(B)$.

$$\otimes\left(\prod_{(u,v)\in\Sigma^k\times\Sigma^l}^{\odot}\mu(v)^{\odot p(uv)}\right)$$

$$\text{(by 6.2.1,1)}$$

$$= (\mathbb{E}[\mu(w_k)])\otimes(\mathbb{E}[\mu(w_l)])$$

since for all u,

$$\prod_{v\in\Sigma_l}^{\odot}\mu(u)^{p(uv)}=\mu(u)^{\sum_v p(uv)}=\mu(u)^{p(u)} \quad .$$

■

6.3.2 Lemma *We have for all $k\geq 1$,*

$$\rho(\mathbb{E}[\mu(w_k)])\leq\mathbb{E}\rho(\mu(w_k)) \quad .$$

In other words, the $(\max,+)$-spectral radius is a convex map.

Proof We have:

$$\begin{aligned}\rho(\mathbb{E}[\mu(w_k)]) &= \rho(\prod_{w\in\Sigma^k}^{\odot}\mu(w)^{\odot p(w)})\\ &\leq \bigotimes_{w\in\Sigma^k}\rho(\mu(w))^{p(w)}=\mathbb{E}\rho(\mu(w_k))\end{aligned}$$

(by Lemma 6.2.1,2,3). ■

6.3.3 Lemma *For all $k,q\geq 1$, we have*

$$\rho(\mathbb{E}[\mu(w_{k\times q})])\geq(\rho(\mathbb{E}[\mu(w_q)]))^k \quad .$$

Proof Via Lemma 6.3.1 and 6.2.2,

$$\rho(\mathbb{E}(\mu(w_{k\times q})))\geq\rho((\mathbb{E}[\mu(w_q)])^k)=(\rho(\mathbb{E}[\mu(w_q)]))^k \quad .$$

■

By Lemma 6.3.3, we have

$$\begin{aligned}\rho(\mathbb{E}[\mu(w_q)])^k &\leq \rho(\mathbb{E}[\mu(w_{k\times q})])\\ &\leq \|\mathbb{E}[\mu(w_{k\times q})]\|\\ &\leq \mathbb{E}\|\mu(w_{k\times q})\| \quad .\end{aligned}$$

Since the sequence $(\mathbb{E}\|\mu(w_k)\|)^{\frac{1}{k}}$ admits the limit $\mathfrak{l}$, we have

$$\lim_{q\to\infty}\mathbb{E}\|\mu(w_{k\times q})\|^{\frac{1}{k\times q}}=\lim_{l\to\infty}(\mathbb{E}\|\mu(w_l)\|)^{\frac{1}{l}}=\mathfrak{l} \quad .$$

Together with Lemma 6.3.3 (set $q=1$), this proves the first half of Theorem 6.1.1. It remains to show that the limit is attained. We have

$$\begin{aligned}\mathfrak{l}^k &\leq \mathbb{E}\|\mu(w_k)\|\\ &\quad\text{(from (53))}\\ &\leq \Delta(\mu)\mathbb{E}\,|\mu(w_k)|_\wedge\\ &\leq \Delta(\mu)\,|\mathbb{E}[\mu(w_k)]|_\wedge\\ &\leq \Delta(\mu)\rho(\mathbb{E}[\mu(w_k)])\end{aligned}$$

Hence, for all $k \geq 1$,

$$(\rho(\mathbb{E}[\mu(w_k)]))^{\frac{1}{k}} \leq \iota \leq (\Delta(\mu)\rho(\mathbb{E}[\mu(w_k)]))^{\frac{1}{k}} \ .$$

We are done. ∎

Concluding remarks

In this paper, we have introduced the notion of Timed Automata as a natural extension of deterministic timed event graphs. We have provided some characterizations of the worst case, optimal case and mean case performance. We have the feeling that from the practical point of view, the most useful result is the simplest mathematically, i.e. Proposition 3.1.2 which provides an $O(n^3)$ algorithm for the worst case analysis. The remaining part of the paper seems to us far more interesting from the algebraic point of view, and its practical relevance is clear, but it suffers of a greater complexity. There are some special cases in which certain further algebraic developments allow computing more quickly the Lyapunov exponents [16].

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